Some Results in Transcendental Number Theory

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Introduction

In the mid-1800's, Liouville published a paper in which he had obtained numbers which satisfied no algebraic equation over the integers. The property of a number being irrational, however, dates back even further. Euler, in 1744, proved the irrationality of e while in 1761 Lambert proved that π was irrational.

It is true that there exist irrational numbers, α, such that $f(\alpha) \neq 0$, $\forall f(x) \in \mathbb{Z}[x]$. However, there is much more to investigate in the above statement. We wish for there to be criteria to determine to which numbers the above fact applies. Moreover, some techniques of approximating irrational numbers by rational numbers lead to methods for proving a number is not a root of any polynomial..

This paper will define some basic terminology of algebraic and transcendental numbers, present some of the above methods as well as discuss and prove the major results regarding e, π and combinations thereof. We finish the paper by stating the Gelfond-Schneider Theorem of 1934 which powerfully classifies a large collection of numbers as being transcendental.

Irrationality Leading to Transcendence

We start off by the defining the key terms for our discussion.

Definition 1. A *number field*, K, is any finite extension of the field of rational numbers $\mathbb{Q}$.

Definition 2. Let K be a number field. An element $\alpha \in K$ is called *algebraic* if there is a polynomial $f(x) \in \mathbb{Z}[x]$ of degree at least one such that $f(\alpha) = 0$. We say that the *degree of* α is the degree of the polynomial $f(x)$ of least degree such that $f(\alpha) = 0$.

Definition 3. A element β is said to be *transcendental* if, for all $f(x) \in \mathbb{Z}[x]$, $f(\beta) \neq 0$. i.e. β is transcendental if it is not algebraic.

As mentioned above, Euler and Lambert proved the irrationality of e and π but more important to the theory of transcendental numbers is a relationship between the properties of irrationality and transcendence. It is known that any irrational number can be approximated using rational numbers. However, in 1844 Liouville observed that for algebraic numbers of degree greater than one, i.e. irrational algebraic numbers, there is a limit to the accuracy with which they can be approximated.

Theorem 4. (Liouville's Estimate) For any algebraic number α of degree $n > 1$, there exists $c = c(\alpha)$ such that $|\alpha - \frac{p}{q}| > \frac{c}{q^n}$ for all $\frac{p}{q} \in \mathbb{Q}$ $(q > 0)$

Proof. Since α is algebraic, there is an irreducible polynomial P of degree $n > 1$ such that $P(\alpha) = 0$. Thus, by the Mean Value Theorem we have:

$$-P(\frac{p}{q}) = P(\alpha) - P(\frac{p}{q}) = (\alpha - \frac{p}{q})P'(\xi) \tag{1}$$

for some ξ between α and $\frac{p}{q}$.

Now if $|\alpha - \frac{p}{q}| \geq 1$, then we can set $c = 1$ so that $n > 1$ and $q \neq 1$ imply that $|\alpha - \frac{p}{q}| > \frac{c}{q^n}$ for all possible $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$. Therefore, we assume $|\alpha - \frac{p}{q}| < 1$ so that $|\xi| < 1 + |\alpha|$. As a result, there is some $c = c(\alpha) > 0$ such that $|P'(\xi)| < \frac{1}{c}$. Thus

$$|\alpha - \frac{p}{q}| > c\,|P(\frac{p}{q})| \tag{2}$$

Now if $P(\frac{p}{q}) = 0$ then that would violate the fact that $P(x)$ is irreducible. Thus $P(\frac{p}{q}) \neq 0$ and so $|q^n P(\frac{p}{q})| \in \mathbb{N}$. Therefore, $q^n|\alpha - \frac{p}{q}| > c|q^n P(\frac{p}{q})| \geq c$ and thus $|\alpha - \frac{p}{q}| > \frac{c}{q^n}$ as desired.

Thanks, [BA] □

With his Estimate, Liouville established a class of transcendental numbers, appropriately termed Liouville Numbers, which satisfy the contrapositive of his Estimate. Two examples of Liouville Numbers are $\omega = \sum_{k=1}^{\infty} 10^{-n!}$ and $\xi = \frac{1}{10^{1!}+} \frac{1}{10^{2!}+} \frac{1}{10^{3!}+\ldots}$. Note that ξ is in the notation for a continued fraction.

Unfortunately, Liouville's Estimate does not imply the transcendence of either π or e and thus we require a different approach. However, proofs of the transcendence of e and π were provided as corollaries to a result sketched by Lindemann in 1882 and later proved rigorously by Weierstrass.

What we will do in the next section is to use calculus to prove the transcendence of e and then later derive the transcendence of π as a corollary to the Lindemann-Weierstrass Theorem.

The Transcendence of e and π.

Theorem 5. e is transcendental

Proof. We will use the notation $f^{(i)}(x) := \frac{d^i x}{dx^i}(f(x))$. Now if $f(x) \in \mathbb{R}[x]$ has degree r and we define $F(x) = f(x) + f^{(1)}(x) + f^{(2)}(x) + \cdots + f^{(r)}(x)$, then a simple calculation shows that $\frac{d}{dx}(e^{-x}F(x)) = -e^{-x}f(x)$.

For $k \in \mathbb{N}$, the Mean Value Theorem on $[0, k]$ implies that $e^{-k}F(k) - F(0) = -e^{-\theta_k k} f(\theta_k k)k$ for $\theta_k \in (0, 1)$. If we multiply this equation by e^k, we see that $F(k) - e^k F(0) = -e^{(1-\theta_k)k} f(\theta_k k)k$ and thus we make the following definitions for $i = 1, \ldots, n$:

$$\varepsilon_i := F(i) - e^i F(0) = -ie^{i(1-\theta_i)} f(i\theta_i) \tag{3}$$

Now if we suppose to the contrary that e is algebraic of degree n, then there exist $c_i \in \mathbb{Z}$, $c_o > 0$ such that e satisfies a relation of the form

$$c_n e^n + c_{n-1} e^{n-1} + \cdots + c_1 e + c_o = 0 \tag{4}$$

If we multiply ε_i by c_i for $i = 1, \ldots, n$ and add, we get $c_1 F(1) + c_2 F(2) + \cdots + c_n F(n) - F(0)\big(c_n e^n + c_{n-1} e^{n-1} + \cdots + c_1 e\big) = c_1 \varepsilon_1 + \cdots + c_n \varepsilon_n$ which then simplifies to

$$c_o F(0) + c_1 F(1) + \cdots + c_n F(n) = c_1 \varepsilon_1 + \cdots + c_n \varepsilon_n \tag{5}$$

Note that the above discussion was independent of the choice of $f(x)$. We now specify an $f(x)$ which will help us arrive at a contradiction to the assumption that e is algebraic. Let

$$\begin{aligned} f(x) &= \frac{1}{(p-1)!} x^{p-1} \prod_{i=1}^{n} (i - x)^p \qquad (6) \\ &= \frac{(n!)^p}{(p-1)!} x^{p-1} + \frac{a_o x^p}{(p-1)!} + \frac{a_1 x^{p+1}}{(p-1)!} + \cdots \end{aligned}$$

for a chosen prime $p > n$ and $p > c_o$ and $a_i \in \mathbb{Z}$. We see from this when $i \geq p$, $f^{(i)}(x) \in (p\,\mathbb{Z})[x]$ and thus $f^{(i)}(j) \in p\,\mathbb{Z}$ for all $i \geq p$ and all $j \in \mathbb{Z}$.

By its definition, $f(x)$ has roots $x = 1, 2, \ldots, n$ each with multiplicity p. So for $j = 1, \ldots, n$, $f(j) = 0 = f^{(1)}(j) = \cdots = f^{(p-1)}(j)$. Since $F(j) = f(j) + \sum_{i=1}^{r} f^{(i)}(j)$, we see $F(j) \in p\,\mathbb{Z}$ for $j = 1, \ldots, n$.

Also, $x = 0$ is a root of $f(x)$ of multiplicity $p - 1$ and thus $f(0) = f^{(i)}(0) = 0$ for $i = 1, \ldots, (p-2)$. Now if $i \geq p$, $f^{(i)}(0) \in p\,\mathbb{Z}$ and so we compute $f^{(p-1)}(0)$. We see that $f^{(p-1)}(0) = (n!)^p \in \mathbb{Z}$, however $f^{(p-1)}(0) \notin p\,\mathbb{Z}$ since p is a prime number bigger than n. Since $F(0) = f(0) + \sum_{i=1}^{r} f^{(i)}(0)$ we see that $p \mid F(0)$.

Overall, we see since $p > c_o > 0$ and $p \mid F(0)$, we know that p does not divide the integer $\sum_{j=0}^{n} c_j F(j)$. Recall that $\sum_{j=0}^{n} c_j F(j) = \sum_{i=1}^{n} c_i \varepsilon_i$ and we had defined $\varepsilon_i := \frac{1}{(p-1)!}\left(-ie^{i(1-\theta_i)}(i\theta_i)^{p-1}(1-i\theta_i)^p \cdots (n-i\theta_i)^p\right)$, where $\theta \in (0,1)$. Thus we estimate

$$|\varepsilon_i| \leq \frac{e^n n^p (n!)^p}{(p-1)!} \tag{7}$$

and so we finally see that $\lim_{p\to\infty} |\varepsilon_i| = \lim_{p\to\infty} \frac{e^n n^p (n!)^p}{(p-1)!} = 0$.

Therefore, there exists a prime p large enough so that $p > \max\{c_o, n\}$ so that

$$|c_1\varepsilon_1 + \cdots + c_n\varepsilon_n| < 1. \tag{8}$$

But $\sum_{i=1}^{n} c_i \varepsilon_i = \sum_{j=0}^{n} c_j F(j) \in \mathbb{Z}$ and thus such a large prime p would force the equation $\sum_{j=0}^{n} c_j F(j) = 0$. This, however, is a contradiction because $p \mid 0$ implies $p \mid \sum_{j=0}^{n} c_j F(j)$ which we know cannot be true from above. Therefore e must be transcendental.

Thanks, [HE] ☐

Lindemann's Theorem

Lindemann's Theorem, or actually the Lindemann-Weierstrass Theorem since Weierstrass proved it, is a powerful demonstration that exponential functions are algebraically independent. We shall refer to this theorem in the future, after we prove it, by the notation LW.

Theorem 6. (Lindemann-Weierstrass) For any distinct algebraic $\alpha_1, \ldots, \alpha_n$ and non-zero algebraic numbers $\beta_1, \ldots, \beta_n$, we have

$$\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} \neq 0$$

Proof. We will first introduce some notation and a helpful function. For a polynomial $f(x) = \sum a_k x^k$ we define $\overline{f}(x) := \sum |a_k| \, x^k$. We then define the following function, for $f(x) \in \mathbb{R}[x]$:

$$\begin{aligned} I(t) &= \int_0^t e^{t-u} f(u) du \\ &= e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t) \end{aligned} \tag{9}$$

The second representation is a result of repeated integration by parts. Combining these two definitions we get the inequality

$$\left|I(t)\right| \leq \int_0^t \left|e^{t-u} f(u)\right| du \leq |t| \, e^{|t|} \, \overline{f}(|t|). \tag{10}$$

Now suppose the theorem were false. Then for some positive integer n there exist $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ satisfying the hypotheses so that

$$\beta_1 e^{\alpha_1} + \cdots + \beta_n e^{\alpha_n} = 0 \tag{11}$$

We can assume that $\beta_i \in \mathbb{Z}$ as a result of multiplying the above equation by all conjugates of all of the β_i and then clearing denominators. We can also assume that for each $t = 1, \ldots, n$ there exist integers $0 = n_0 < n_1 < \cdots < n_r = n$ so that $\alpha_{n_t+1}, \ldots, \alpha_{n_{t+1}}$ is a complete set of conjugates for α_t and $\beta_{n_t+1} = \cdots = \beta_{n_{t+1}}$.

Since α_i is algebraic for all i, $\alpha_1, \ldots, \alpha_n$ is a collection of (not necessarily all) roots of some polynomial of degree N over the integers. We let $\alpha_{n+1}, \ldots, \alpha_N$ denote the remaining $N - n$ roots. Thus, for $\beta_{n+1} = \cdots = \beta_N = 0$ we have the following equation:

$$\prod_{(k_1, \ldots, k_n)} \left(\beta_1 e^{\alpha_{k_1}} + \cdots + \beta_N e^{\alpha_{k_N}}\right) = 0 \tag{12}$$

where $(k_1, \ldots, k_n)$ runs over all permutations of $(1, \ldots, N)$.

Now let $l \in \mathbb{N}$ such that $l\alpha_1, \dots, l\alpha_n$ and $l\beta_1, \dots, l\beta_n$ are all algebraic integers and define, for all $i = 1, \dots, n$ $f_i(x) = l^{np}\frac{(x-\alpha_1)^p \cdots (x-\alpha_n)^p}{(x-\alpha_i)}$ for a large prime p.

We let $J_i = \beta_1 I_i(\alpha_1) + \cdots + \beta_n I_i(\alpha_n)$ with $I_i(t)$ defined as in (9) using $f = f_i$. Then some computation shows that for $m = np - 1 = \deg(f_i)$,

$$J_i = -\sum_{j=0}^{m} \sum_{k=1}^{n} \beta_k f_i^{(j)}(\alpha_k) \tag{13}$$

We now note that $p! \mid f_i^{(j)}(\alpha_k)$ unless $j = p-1$ or $k = i$. In that case we see that

$$f_i^{(p-1)}(\alpha_i) = l^{np}(p-1)!\prod_{k \neq i}^{n}(\alpha_i - \alpha_k)^p \tag{14}$$

which is divisible by $(p-1)!$ but not $p!$ Thus $(p-1)! \mid J_i$. Further

$$J_i = -\sum_{j=0}^{m} \sum_{t=0}^{r-1} \beta_{n_{t+1}} \left(f_i^{(j)}(\alpha_{n_t+1}) + \cdots + f_i^{(j)}(\alpha_{n_{t+1}}) \right) \tag{15}$$

and the internal sum is a polynomial with rational coefficients in the α_i. Also the coefficients of $f_i^{(j)}(x)$ can be expressed as rational numbers. Therefore the product $\mid J_1 \cdots J_n \mid \in \mathbb{Q}$ and moreover is divisible by $\left((p-1)!\right)^n$. Thus $|J_1 \cdots J_n| \geq (p-1)!$. However, inequality (2) gives us

$$\mid J_i \mid \leq \sum_{k=1}^{n} \mid \alpha_k \beta_k \mid e^{|\alpha_k|} \overline{f}\left(|\alpha_k|\right) \leq c^p \tag{16}$$

for some $c \in \mathbb{R}$ independent of p. But this contradicts $|J_1 \cdots J_n| \geq (p-1)!$ when the prime p is chosen sufficiently large. Thus the theorem must be true.

Thanks, [BA] □

As a result of this powerful theorem, we can prove in a short corollary the transcendence of π.

Corollary 7. π is transcendental

Proof. Assume that π is algebraic. Then so too must be πi and $2\pi i$. Thus, we consider the special case where $n = 2$, $\alpha_1 = \pi i$, $\alpha_2 = 2\pi i$, and $\beta_1 = \beta_2 = 1$. We derive a contradiction using LW and the following equation:

$$0 \neq \beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} = e^{\pi i} + e^{2\pi i} = -1 + 1 = 0 \tag{17}$$

Therefore , the assumption that π is algebraic must be false.

□

LW reaches even deeper than this result. We can add to our collection of transcendental numbers in another corollary to LW.

Corollary 8. For any algebraic $\alpha \neq 0$, $\cos(\alpha)$, $\sin(\alpha)$, *and* $\tan(\alpha)$ *are all transcendental. Moreover, for algebraic* α *not* 0 *or* 1, $\log(\alpha)$ *is also transcendental.*

The Gelfond-Schneider Theorem

To conclude this treatment of transcendental number theory, we state and describe the implications of the Gelfond-Schneider Theorem.

Theorem 9. (Gelfond-Schneider) If $\alpha_1, \ldots, \alpha_n$ *are nonzero algebraic numbers such that* $\log(\alpha_1), \ldots, \log(\alpha_n)$ *are linearly independent over the rationals, then* $1, \log(\alpha_1), \ldots, \log(\alpha_n)$ *are linearly independent over the field of all algebraic numbers.*

This idea was researched and proved by Gelfond and Schneider independently in 1934. The main results of this theorem are that α^β is transcendental for any algebraic $\alpha \neq 0, 1$ and any imaginary quadratic irrational

β. As a result we know that $e^{\pi} = (-1)^{-i}$ is transcendental. The remaining implications of the Gelfond-Schneider Theorem will be stated as corollaries.

Corollary 10. Any non-vanishing linear combination of logarithms of algebraic numbers with algebraic coefficients is transcendental.

Corollary 11. $e^{\beta_0}\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n}$ *is transcendental for any nonzero algebraic numbers* $\alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n$.

Corollary 12. $\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n}$ *is transcendental for any algebraic numbers* $\alpha_1, \ldots, \alpha_n$ *other than* 0 *or* 1*, and any algebraic numbers* $\beta_1, \ldots, \beta_n$ *with* $1, \beta_1, \ldots, \beta_n$ *linearly independent over the rationals.*

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